INCLUSIVENESS AND EXCLUSIVENESS CASES IN SET THEORY EXPERIMENTATIONS

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ABSTRACT

The concept of set theory is widely used in our everyday activities. The idea of subsets is to play safe while experimenting with the elements of the parent set (also called universal set, 'U'), to ensure that each subset is distinct from others (by inclusiveness or exclusiveness of one or more of the elements. The elements are uniquely identified from one another, even when they have at least one common characteristic. In this work, the author constructed models used in handling cases of inclusiveness and exclusiveness of some elements of the universal set in the subsets. Considering the total number of subsets as a sample space of subsets of U, the author also introduced probability models to represent the number of required subsets.

Keywords: universal set;subsets; empty set; inclusiveness; geometric progression; exclusiveness

1. Introduction

Set theory is a branch of mathematics that deals with the properties of a well-defined collection of objects. In the last updated Article History, June 28, 2024, the Editors of Encyclopaedia, Britannica, stated that set theory was dated to between 1874 and 1897 when a German Mathematician, Georg Cantor, created a theory of abstract sets of entities and made it into a mathematical discipline. A lot of criticism followed and by 1900, the Cantor's ideas and results were fully recognized as a distinct branch of mathematics. So many theorems have been developed by great mathematicians. Today, almost all aspects of human activities involve applications of concepts of set theory. Experimenting with sets involves using the subsets of the parent set (also called Universal set). It is conventional that given universal set 'U' with 'n' number of elements., the number of subsets is given by $N = 2^n$. These subsets form a sample space from where all results of associated experiments are selected. The author in this work intends to introduce some models to enhance easy applications of the concepts associated with set theory.

2. Discussions

2.1 Generalized Inclusive Case of Elements of a Non-Empty Set in the Subsets

We recall that given a non-empty set U with 'r' elements, we write that $n(U) = r$. The number of subsets of U is denoted by N and given by

$N = 2^r$, always even number. (1)

The author of this article wishes to construct Mathematical models that answer the questions of "How many of the subsets of U contain the identified elements of U?"

Let 'k' be the number of some identified elements in U that must be included in the required subsets. We shall adopt the principle of mathematical induction to lead us to the generalized case.

i. Let n(U) = r = 2. Then, $N = 2^2 = 4$ possible subsets (including U itself as well as the empty set.Suppose U = {a, b}, such that 'a' is identified/ chosen to be in any required subset. Write N_a to represent the number of subsets of U that has 'a' as one of its elements. With subsets of U as {a, b}, {a}, {b} and { }. We observe that two subsets contain 'a'. We write $N_a = 2 = 4/2 = N/2^1$, for k = 1.

ii. Increase number of elements in U to have U = {a, b, c}. with r = 3, we have $N = 2^3 = 8$ subsets. The subsets are {a, b, c}, {a, b}, {a, c}, {b, c}, {a}, {b}, {c} and{ }. Observe that four of the subsets contain the element 'a'.Mathematically, we have $N_a = 2^r / 2^k = 2^3/2^1 = 4$.

Suppose we identify/ choose "a and b". Then, k = 2 to get $N_{a,b} = 2^r/2^k = 2^{r-k} = 2^{3-2} = 2^1 = 2$ subsets. Suppose we identify/ choose "a, b and c". With r = 3 and k =3, it is trivial that only one subset contains all elements of U (U is a subset of itself).Thus, $N_{a,b,c} = N_{all} = 2^{3-3} = 2^0 =1$ subset.

iii. Take U = {a, b, c, d}, r = 4. The subsets of U are {a,b,c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a}, {b}, {c}, {d} and { }. Suppose 'a' is chosen as before, then k = 1. Thus $N_a = 2^{r-k} =2^{4-1} =2^3 = 8$ subsets.
Suppose we choose 'a and b', k=2, gives $N_{a,b} = 2^{4-2} = \ldots = 4$ subsets.
Suppose we choose 'a, b and c', k=3, gives $N_{a,b,c} = 2^{4-3} = \ldots = 2$ subsets.
Suppose we choose 'a, b, c and d', k=4, gives $N_{a,b,c,d} = N_{all} = 2^{4-4} = 1$ subset.

iv. Take U {a,b,c,d,e}, r = 5. The number of subsets of U = $N = 2^5 = 32$.
The subsets of U are {a,b,c,d,e}, {a,b,c,d}, {a,b,c,e}, {a,b,d,e}, {a,c,d,e}, {b,c,d,e}, {a,b,c}, {a,b,d}, {a,b,e}, {a,c,d}, {a,c,e}, {a,d,e}, {b,c,d}, {b,c,e}, {b,d,e}, {c,d,e}, {a,b}, {a,c}, {a,d}, {a,e}, {b,c}, {b,d}, {b,e}, {c,d}, {c,e}, {d,e}, {a}, {b}, {c}, {d}, {e}, { }.
Suppose 'a' is chosen as before, then k = 1. Thus $N_a = 2^{r-k} =2^{5-1} =2^4 = 16$.
Suppose we choose 'a and b', k=2, gives $N_{a,b} = 2^{5-2} = \ldots = 8$ subsets.
Suppose we choose 'a, b and c', k=3, gives $N_{a,b,c} = 2^{5-3} = \ldots = 4$ subsets.
Suppose we choose 'a, b, c and d', k=4, gives $N_{a,b,c,d} = N_{all} = 2^{5-4} = 2$ subsets.
Suppose we choose 'a, b, c, d and e', k=5, gives $N_{a,b,c,d,e} = N_{all} = 2^{5-5} = 1$. subsets.

v. At this point, we wish to write down a general model.
Take $U = \{a_j, j = 1,2,3,\ldots,n\}, n\geq 2$. (2)
The number of subsets of U is $N = 2^n$. See equation (1).
If we identify 'k' objects from U, to ensure they are contained in some subsets of U. The number of such subsets is denoted by N_k, and given by
$N_k = 2^{n-k}, (k\geq 1, n\geq k, 1 \leq k \leq n)$. (3)

Remark 1: The author makes some observations supporting equation (3) above. Given set U with 'n' objects and number of subsets N, given by $N = 2^n$. Suppose we wish to identify 'k' objects from U, to ensure that they are contained in some subsets of U. The number of such subsets is N_k, given by $N_k = 2^{n-k}$. We remark that N_k diminishes as 'k' increases such that the ratio of
"N_k to N_{k+j}" $= 2^{k+j} : 2^k$, for $k = 1, 2, 3, \ldots, n$; $j = 0,1,2, \ldots, n-1$ and $k+j \leq n$. (4)

To support this, consider example iv above. U has n = 5 elements, with $N = 2^5 = 32$ subsets. For the 'k' identified number of objects as before, we have that $N_1 = 16$; $N_2 = 8$; $N_3 = 4$; $N_4 = 2$ and $N_5 = 1$. Observe that the N's form a **geometric progression, GP** given by 32, 16, 8, 4, 2 and 1.First term, **a** = 32, common ratio, **r** = ½ and nth term$\mathbf{ar^{n-1}}$. The other observation is that while $N = 2^n$ subsets, the last term given by N_{all} is always 1 subset, Since U is a subset of itself.

Remark 2: Practical Benefit of the New Model

Given U with 'n' objects, such that $N = 2^n$ = number of subsets of U. The mathematics of calculating N and the N_k's is always easy. However, it could be cumbersome to list all the subsets before selecting the required ones if we show interest on some of the elements of U (using k as above). We hereby suggest the use of **exclusion** method and the concept of **union** of sets/ subsets.

First: Identify all the 'k' objects of interest. Fish them out from the U. with this exclusion, we are left with 'n-k' objects to experiment with.

Secondly: Show the sample space of the possible subsets using the remaining 'n-k' objects of U.

Thirdly: Suppose '**K**' is the set of all the identified '**k**' objects from U. Seek the **union** of '**K'** and all the subsets of **U***, where **U*** is the sample space of all subsets of **U** without '**K**'.This new sample space answers the question of "show the '2^{n-k}' subsets of U that must contain the identified 'k' objects of U".

By this approach, the author believes to have constructed models capable of ensuring reduced resource inputs for optimum results.

Example 2.1

Given U = {a, b, c, d, e}. Then n = 5 implies number of subsets of U is $N = 2^5 = 32$. Suppose 'a, b and c' are identified to be in some of these subsets of U. Mathematically, k = 3, yields $N_3 = 2^{5-3} = 2^2 = 4$ subsets. Truly, enlisting 4 subsets is more economical than enlisting 32 subsets. Thus, U* = {d,e}, with subsets given as {d,e}, {d}, {e} and { }. But K = {a, b, c}. Finally, the union of K and all subsets of U* are given by **S** = {a, b, c, d, e}, {a, b, c, d}, {a, b, c, e} and {a, b, c}.

Example 2.2

Enugu State has 10 ministries in the civil service system. A committee has been set up bythe Government to make recommendations for the reappointment / fresh appointment of 10 Commissioners. Government is showing special interest in 3 ministries "A, B and C" forreappointment. How many ways can this project be achieved now, bearing in mind thatthe Government has not been comfortable with the cost of using the commissioners? To solve this problem, we note that Government can undertake the reappointment / appointment based on economy of the State. Take n = 10, given U = {a, b, c, …, j}. Thus, $N = 2^n = 2^{10} = 1024$ subsets (possibilities). With k = 3, the required number of subsets is $N_k = 2^{n-k} = 2^7 = 128$ subsets. To show these 128 subsets, K = {a, b, c} and U* = { d, e, f, g, h, i, j}. the union of K and all subsets of U*, gives the required sample space, S = {a,b,c,d,e,f,g,h,i,j}, **…**, {a,b,c,d}, {a,b,c,e}, {a,b,c,f}, {a,b,c,g}, {a,b,c,h}, {a,b,c,i}, {a,b,c,j} and {a,b,c}. Note that the gap is for the other **119** subsets. In all, it is more economical to enlist and accommodate 128 subsets of U, than the entire 1024 subsets.

2.2 Generalized Exclusive Case of Elements of a Non-Empty Set from the Subsets

The greater part of the job is already done in the inclusive case. Given U, with n elements. Then, $N = 2^n$ is the number of subsets of U. if 'k' elements of U are identified to be excluded from the subsets, the required number is N_{exc}, given by $N_{exc} = N – N_{inc} = 2^n – 2^{n-k}$ subsets, provided $k \leq n$.
(5)

Note that we have used N_{inc} for N_k, where the subscript 'inc' standsfor inclusive.

Remark 3: Probability and Sets

The author wishes to show that the knowledge of sets and subset can be applied in solving discrete probability problems. In this case, the sample space contains finite number of sample events. We shall take the subsets of U to be the sample events.

Example 3.1

Let U = {a, b, c, d, e}. With n = 5, we have number of subsets of U as $N = 2^5 = 32$. These were seen in example 1(iv).

i. Observe that U is a subset of itself. Probability of all the elements chosen as subset = 1/32.

ii. Five subsets have 4 elements each. Prob (chosen subset has 4 elements) = 5/32.

iii. Ten subsets have 3 elements each. Prob (chosen subset has 3 elements) = 10/32.

iv. Ten subsets have 2 elements each. Prob (chosen subset has 2elements)= 10/32.

v. Five subsets have 1 element each. Prob (chosen subset has 1 element) = 5/32.

vi. The only subset with no element is the empty set. Its probability = 1/32.

vii. Prob ('a' must be included in any subset) = 16/32.

viii. Prob ('a and b' must be included in any subset) = 8/32.

ix. Prob ('a, b and c' must be included in any subset) = 4/32.

x. Prob ('a, b, c and d' must be included in any subset) = 2/32.

xi. Prob ('a, b, c, d and e' must be included in any subset) = 1/32.

xii. Observe that the sum, 1/32 + 16/32 +8/32 +4/32 + 2/32 + 1/32 = 1, total probability. Also, the sum, 1/32 + 5/32 + 10/32 + 10/32 + 5/ 32 + 1/32 = 1.

[Again, ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$, ${}^5C_5$ = 1, 5, 10, 10, 5, 1. This is kept for other considerations, involving combinations].

Example 3.2:

A company has 'n' cities under its watch to introduce its product. There is alternative in case some cities are not visited in this period.

i. How many ways can the cities be visited in this period?
ii. If 'k' cities have been identified to be included in the visits, how many possibilities does the company have?

iii. If ‘y’ cities are excluded in this period’s visits, how many possibilities does the company have?

iv. What is the probability that ‘k’ cities are among those visited?

v. What is the probability that ‘y’ cities were excluded during the visits?

Note that $k \leq n$, and $y \leq n$ to ensure feasibility of the solutions.

Solutions

Observe that this is an example of General Question. It can apply to the following:

- Travelling salesman, who visits ‘n’ cities in a tour/ period, starting and ending at city 1, all things being equal.
- Visit every community programme by the Government in a period, with total of ‘n’ communities, all things being equal.
- Visit every Local Government Area of a state by the Governor in a period, all thing being equal.
- Visit every faculty of an Institute by the Rector in a period, all things being equal.
- Rehabilitate the ‘n’ rural roads in a period, by the concerned authority, all things being equal.

To be particular, let n=5, k= 3 and y=2, as in example 3.2 above.

i. Take U = {a, b, c, d, e} as the set of the cities to be visited. Number of ways the cities may be visited in the period is just the number of possible subsets of U given by $N = 2^n = 2^n = 32$ ways.

ii. Take ‘a, b, c’ as the k=3 specially favoured cities. Then, $N_k = 2^{n-k} = 2^{5-3} = 4$ ways.

iii. Take ‘c and d’ as the y = 2 excluded cities. But inclusive y = 2 cities $= N_2 = 2^{n-2} = 2^{5-2} = 8$ ways. Then $\not{N}_2 = 2^n - 2^{n-y} = 32 - 8 = 24$ ways.

iv. Probability (that ‘a, b, c’ as the k=3 cities favoured) $= N_k/N = 2^{n-k} / 2^n = \ldots = 2^{-k} = 2^{-3} = 1/8 = 0.125$. [Note: 4/32 = 1/8 = 0.125 also].

v. Probability (that ‘c and d’ as the y=2 cities excluded) $= \not{N}_y/N = (2^n - 2^{n-y})/2^n = 1-2^{-2} = 1- ¼ = ¾$. [Note: 24/32 = 0.75].

Let’s finally observe that the subset {a, b, c, d} includes c and d as elements alongside a and b. However, the subset {a, b, c, e} does not include c **and** d. It includes c but excludes d.

3. Summary and Conclusion

This work has shown that set theory concepts are applied in our daily activities. The subsets are used to describe the parent set (also called universal set}. The author constructed models that answer the questions of “How many subsets that include some ‘k’ objects of U, on one side, and How many subsets that exclude ‘y’ objects of U, on the other hand”. The models are displayed in equations (4) and (5). He noted that given U, with $N = 2^n$ as number of subsets of U. The N_k’s (k = 1, 2, …, n) and N form a **geometric progression** with first term as, $\mathbf{a} = N = \mathbf{2^n}$, common ratio $\mathbf{r = ½}$ and $\mathbf{n^{th}}$ term as $\mathbf{ar^{n-1}}$, to yield last term as $N_{all} = 1$ subset.

Considering the subsets as sample events, the author introduced probability concepts to solving set theory problems. He proposed using the combinations concepts to solving set theory problems, as further research works. In fact, other applications of set theory need to be established.

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