GENERALIZED INCLUSIVENESS AND EXCLUSIVENESS CASES OF PERMUTATIONS AND COMBINATIONS

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ABSTRACT

Scientific experimentations involve following some procedures to work with the available resources to achieve optimum results. In our everyday activities, the methods of counting of objects are provided in the concepts of permutations and combinations. Scientists and in particular, mathematicians and statisticians have done well by providing updated models used in solving permutation and combinatorial problems. "Generalized Inclusive Case of Permutations and Generalized Exclusive Case of Permutations". The Experimental Descriptions for the two models are also provided to enhance easy applications

Keywords: permutation; inclusiveness; exclusiveness

1. Introduction

The vast applications of permutations and combinations call for attention. Some of the existing models need to be strengthened. Most of our daily activities require counting of objects, whether similar or not. This is the basis of permutations and combinations which have attracted so many models and updates from scientists.

The vast applications of permutations and combinations find favour in their special cases including when one of the 'n' objects meant for the experiment, should be included in (or excluded from) any 'r' selected at a time. Other operations favour selecting all the 'n' objects at a time. This work is intended to provide models that will serve as generalized approach to such experiments.

2. Discussions on Permutations and Combinations

Scholars and researchers have done much in the updated discussions on permutations and combinations. Gregersen (2024), said that both concepts explain the various ways in which objects from a set may be selected without replacement to form subsets. When the order of selection is a factor, it is called a permutation. It is called a combination when the order of selection is not a factor. He recognized that the concepts came to be when in the 17th century, the French Mathematicians Blaise Pascal and Pierre de Fermat developed combinatorics and probability theory. The formulae remain

$$^{\mathbf{n}}\mathrm{P}_{\mathbf{r}} = \frac{n\,!}{(n-r)!} \qquad (1)$$

where $^{\mathbf{n}}\mathrm{P}_{\mathbf{r}}$ represents the permutation of n objects, taken r at a time; n! (read as "n factorial") is the product of the "n" consecutive whole numbers starting from 1.

Generally, $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$. Note that $1! = 1$ and $0! = 1$ also.

For combinations, we write $^{\mathbf{n}}\mathrm{C}_{\mathbf{r}}$ to represent the combination of n objects, taken r at a time. It is given by

$$^{\mathbf{n}}\mathrm{C}_{\mathbf{r}} = \frac{n\,!}{(n-r)!r!} \qquad (2)$$

Generally, it is believed that both formulae represent the ratio of "number of required selections to the number of possible subsets". Childress (1974), generalized that to select r objects at a time from a given set of n objects, the order of selection may or may not matter. If the order is necessary, we call it permutation, and the mathematical representation is as in (1) above. However, if the order of selection is not necessary, it is called combination, and is represented as in (2) above.

The basic knowledge in permutations and combinations were reviewed from the books and articles referenced. Permutations and combinations concepts adopt a fundamental rule for counting objects. The rule regards counting objects as experiments. This rule states that: If an experiment is made up of 'k' steps such that first, second, …, k^{th} steps can be done respectively in $n_1, n_2, \ldots, n_k$ ways. Then, the entire experiment has N different possible outcomes, given by

$$N = n_1 \times n_2 \times \ldots \times n_k \text{ ways} \qquad (3)$$

This forms a product of n_k ways.

Example 1

A committee of three (one each of labour leader, Government official and community leader), is to be formed from a group of 3 labour leaders, 5 Government officials and 7 community leaders. We wish to calculate the number of ways of selecting the committee.

To answer this question, we observe three steps to take.

Step 1: There are 3 ways of selecting 1 labour leader from group of 3 ($n_1 = 3$).

Step 2: There are 5 ways of selecting 1 Government official from group of 5 ($n_2=5$).

Step 3: There are 7 ways of selecting 1 community leader from group of 7($n_3 =7$).

Finally, the number of ways of selecting the committee becomes N given by

$N = n_1 \times n_2 \times n_3 = 3 \times 5 \times 7 = 105$ ways.

Remark 1

At this point, it is important to mention that in practice, there are needs to play some interest while experimenting on permutations and combinations problems. These special cases are considered.

Case 1: Inclusiveness Case of Permutations

Given a set of 'n' dissimilar objects. Suppose one of the objects must be included in any 'r' objects selected at a time. The required number of ways of achieving this special goal is N, given by

$$N = N_{inc} = r \times {}^{n-1}P_{r-1} \tag{4}$$

where N_{inc} denotes inclusive case of permutations.

Case 2: Exclusiveness Case of Permutations

Given a set of 'n' dissimilar objects. Suppose one of the objects must be excluded in any 'r' objects selected at a time. The required number of ways of achieving this special goal is N, given by

$$N = N_{exc} = {}^{n-1}P_r \tag{5}$$

where N_{exc} denotes exclusive case of permutations.

Case 3: All Inclusiveness Case of Permutations

Given a set of 'n' objects (not necessarily the same). Suppose we select all objects at a time. The number of ways of doing the experiment is N, given by

$$N = N_{all} = \frac{n!}{f! \times g! \times h!} \tag{6}$$

where N_{all} indicates that all the 'n' objects are selected at the same time. This formula (6) is such that some of the 'n' objects appeared more than once. The details are that one of them appeared 'f' times, another one appeared 'g' times, and the other one appeared 'h' times. Others just appeared once.

At this point, the author reminds us that models (1) to (6) have been in use in solving our permutations problems. It is good to introduce new special cases of permutations.

Remark 2: Generalized Inclusive Case of Permutations

The author considers this as a new special case of permutations:

i. Given a set of 'n' dissimilar objects.
ii. Assume we select 'r' of the 'n' at a time, ($r \leq n$).
iii. Suppose order of selection is necessary such that $ab \neq ba$, given that a and b are some members of the 'n' objects.

iv. Suppose ‘k’ identified objects must be included in any of the ‘r’ selected,provided that ($1 \leq k < r$).

v. The author wishes to present a model that answers the question of how many ways one can do the experiment, with all the interest above.

To achieve this, let N represent the number of ways of selecting ‘r’ objects at a time from a set of ‘n’ dissimilar objects in which ‘k’ identified members of the set must be included in any ‘r’ selected. Then, N is given by

$$N = {}^{r}P_{k} \times {}^{n-k}P_{r-k} \qquad (4^*)$$

We wish to state that equation (4*) is the generalized inclusive case of permutations. Note that expansion of ${}^{r}P_{k}$yields the product:

$${}^{r}P_{k} = r \times (r-1) \times (r-2) \times \ldots \times (r-k+1) \text{ for } (1 \leq k < r). \quad (4^{**})$$

Remark 2.1: Experimental Description of the Generalized Inclusive Case ofPermutations

This experiment is presented in two phases.

Phase 1: Feasibility Assessment

Revisit Remark 2, to ensure that all the conditions attached to n, r and k are satisfied. The experiment can be done, provided that $r \leq n$ and $1 \leq k < r$, which combines to

$$1 \leq k < r \leq n \qquad (4a)$$

Phase 2: To establish equation (4*), that $N = {}^{r}P_{k} \times {}^{n-k}P_{r-k}$.

Step 1: Single out all the ‘k’ objects that must be included in any ‘r’ selected. When this is done, we are left with selecting r-k objects from the remaining n-k. This activity represents ${}^{n-k}P_{r-k}$ as in equation (4*).

Step 2: Because $k<r$, it is true that every r selected must include all the identified k objects. Any one of the identified k objects can assume any position when arranged within themselves and among the other selected r-k from n-k. [If $a_1, a_2, \ldots, a_k$are the ‘k’ identified objects, then $a_1a_2 \neq a_2a_1$, say]. Thus, we are faced with the permutation of ‘r’ objects taken ‘k’ at a time, with the existing conditions attached. This activity yields ${}^{r}P_{k}$ as in equation (4*).

Step 3: Activities in steps 1 and 2 need to be combined. The language is, “undertake step 1 <u>**and**</u> step 2 to achieve phase 2”. The underlined connotes the mathematical operator, “**multiplication**”. When this is done, the experiment is achieved, giving rise to N as in equation (4*).

Example 1

Given U ={a, b, c, d, e}. Select 4 objects at a time. The sample space is shown in Appendix I. From existing model, the permutation of ‘n’ objects, taken ‘r’ at a time is given by $N = {}^{n}P_{r} = {}^{5}P_{4} = 120$ ways.

(i). If ‘a’ must be included in any of the selection. Then n = 5, r = 4, and k = 1.

Thus, $N = {}^{r}P_{k} \times {}^{n-k}P_{r-k} = {}^{4}P_{1} \times {}^{5-1}P_{4-1} = \ldots = 96$ ways.

$[24 + 6 \times 4 + 4 \times 12] = 24 + 24 + 48 = 96$ ways also, from Appendix I.

(ii). If ‘a and b’ must be included in any of the selection. With n = 5, r = 4, and k = 2, we have $N = {}^{4}P_{2} \times {}^{5-2}P_{4-2} = \ldots = 72$ ways.

Note that $[6 \times 2 + 4 \times 12 + 2 \times 6] = 12 + 48 + 12 = 72$ ways also.

(iii). If ‘a, b and c’must be included in any of the selection. With n = 5, r = 4, and k = 3, we have $N = {}^{4}P_{3} \times {}^{5-3}P_{4-3} = \ldots = 48$ ways.

Note that $[4 \times 6 + 2 \times 12] = 24 + 24 = 48$ ways also.

(iv). If ‘a, b, c and d’must be included in any of the selection. With n = 5, r = 4, and k = 4, we have $N = {}^{4}P_{4} \times {}^{5-4}P_{4-4} = \ldots = 24$ ways.

From Appendix I, $[2 \times 12] = 24$ ways also.

Remark 3: Generalized Exclusive Case of Permutations

The statements in this case are related to that of Remark2, such that statements i, ii and iii, still apply.

iv. Suppose ‘k’ identified objects must be excluded from any ‘r’ selected,with the condition that$(k + r \leq n)$.

v. The author hereby presents a model that answers the question of how many ways one can perform the experiment as in i to iv above. Take N as the number of ways required. Then, we have

$$N = {}^{n-k}P_{r}, \text{ given that } k + r \leq n. \quad (5)$$

Note that the condition, $k + r \leq n$, must be satisfied before exclusiveness can hold.

Remark 3.1:Experimental Description of the Generalized Exclusive Case of Permutations

This is also presented in two phases.

Phase 1: Feasibility Assessment.

Revisit Remark 3 to ensure that all conditions attached to n, r, and k are satisfied. This experiment can be achieved, provided that $k+r \leq n$ and $r < n$.

Phase 2

Step 1: Identify the k objects to be excluded. Then fish them out from the whole n objects. At this point, we are left with n-k objects.

Step2: Select the required r objects from the remaining n-k. This activity is represented by ${}^{n-k}P_{r}$ as in Equation (5). At this point, the experiment is achieved.

Example 2

Given U ={a, b, c, d, e}. Select 4 objects at a time. With n = 5, the sample space is shown in Appendix I.

i. If ‘a’ must be excluded from any r selected. Take k = 1. Thus,N = ${}^{n-k}P_r = {}^{5-1}P_4 = \ldots = 24$ ways. Truly, it is obvious that the algebraic complement of exclusiveness is inclusiveness, and vice versa. Refer to example 1 to see that

$^nP_r = {}^rP_k \times {}^{n-k}P_{r-k} + {}^{n-k}P_r$, provided thatk + r ≤ n(5*)

[Lhs = $^5P_4$=120 ways. Rhs= $^4P_1 \times {}^{5-1}P_{4-1} + {}^{5-1}P_4$=96+24= 120 ways also].

ii. If ‘a and b’ must be excluded from any r selected. Here, k = 2. Note that excluding 2 objects from 5 gives a remainder of 3 objects, and the issue of selecting 4 at a time is no longer feasible. This is true because the feasibility condition of equation (5*) fails.

iii. Given the same U as above, with n = 5 dissimilar objects. Let us select 3 objects at a time. Suppose a and b must be included in any of the ‘r’. Then, k = 2. The number of ways of achieving this experiment is N, given by N = $^3P_2 \times {}^{5-2}P_{3-2} = \ldots = 6\times 3 = 18$ ways.

iv. Suppose iii is changed to say that a and b must be excluded in any of the 3 selected. Then n = 5, r = 3, and k = 2 put in equation (5), provides N as N = $^{n-k}P_r = {}^3P_3 = 6$ ways. Note that if no further condition is provided to n and r, then N = $^nPr = {}^5P_3 = 60$ ways. But 60 ≠ (18 + 6 = 24). This shows that equation (5*) fails if k ≥ 2.

To generalize this case of inclusiveness as algebraic complement of exclusiveness becomes a matter for further research.

Finally, we state that the author, in the presentations above did not depart from the basics of permutations and combinations. In all, each N defined and / or calculated still ended as an ‘even number’, not minding that the number of objects used for the experiment may be even or odd.

3. Applications of permutations and Combinations

- Arrangement of seats and sittings for events.
- Creating and assignment of phone numbers.
- Creating and assignment of motor vehicle registration numbers.
- Grouping of football teams and match fixtures for tournaments.
- Selection of candidates for academic and excellence awards.
- Selection of nominees for elective positions.
- Selection of lottery numbers.
- Selection of teams and team leaders for academic competitions.
- Selection of colours for painting works.
- Selection of candidates for employments.
- Selection of sample spaces for statistical experiments.
- Selection of sample spaces for equipment procurement, utilization and replacement.

- Selection of workers for appraisals, workshops, trainings and conferences.
- Selection of textile design patterns.
- Selection and assignment of portfolios to workers.
- Selection of sample spaces for strategies in indoor games.
- Selection of sample spaces of possible tours for the travelling salesman.
- Selection of sample spaces of possible sequences of homogeneous products on factory machines.

4. Summary and Conclusion

We have seen that permutations and combinations techniques are used in counting of objects in a given set. Their applications cut across so many fields of life, as seen in the discussions. This call is necessary especially in this computer and internet age, to ensure maximal utilization of our resources.

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